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Epigraphs of Convex Set Functions

J. H. CHOU, WEI-SHEN HSIA, AND TAN-YU LEE*

Department of Mathematics, University of Alabama, University, Alabama 35486

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In this paper, we characterize a convex set function by its epigraph. The w^*-semicontinuities for set functions are defined. We identify a convex set function with a functional in L_∞ and show that the w^*-closure of such a functional is a convex functional. A Fenchel duality theorem for set functions is then derived.

1. INTRODUCTION

In 1976, a definition of a convex set function was given by Dolecki and Kurcyusz in [1]. Using a more natural approach, Morris [2] defined a convex set function by a special class of sequences which is akin to the concept of "convex combination" of two sets. Following Morris' idea, the definition of a convex subfamily $\mathscr{S}$ of $\mathscr{A}$ was given and properties of convex set functions defined on $\mathscr{S}$ were investigated by the authors in [3].

In Section 3, convex set functions are characterized through their epigraphs. Furthermore, we identify a convex set function on a convex subfamily $\mathscr{S} \subset \mathscr{A}$ with a functional on L_∞ over $\chi_{\mathscr{S}} = \{\chi_\Omega | \Omega \in \mathscr{S}\}$ and show that the w^*-closure of such a functional is a convex functional on the w^*-closed convex hull of $\chi_{\mathscr{S}}$ in L_∞. We also define the w^*-semicontinuities for set functions through the w^*-closureness, and characterize the w^*-continuous convex set functions. A Fenchel duality theorem for set functions is then established through the w^*-closure of convex set functions in Section 4.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, it will be assumed as in [2] that $(X, \mathscr{A}, m)$ is a finite atomless measure space with $L_1(X, \mathscr{A}, m)$ separable. We denote by I

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the interval [0, 1]. For $\Omega \in \mathscr{A}$, χ_Ω denotes the characteristic function of Ω. We shall write L_p instead of $L_p(X, \mathscr{A}, m)$. For $\mathscr{S} \subset \mathscr{A}$, we write $\chi_{\mathscr{S}} = \{\chi_\Omega: \Omega \in \mathscr{S}\}$ and define $\bar{\mathscr{S}} = w^*$-closure of $\chi_{\mathscr{S}}$ in L_∞. In [2] Morris showed that given sets Ω, $\Lambda \in \mathscr{A}$, and $\lambda \in I$, there exist L_∞-sequences

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \sim \Lambda}, \qquad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\, \chi_{\Lambda \sim \Omega}$$

and

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Omega + (1-\lambda)\, \chi_\Lambda,$$

consequently $\bar{\chi}_{\mathscr{A}}$ contains the convex hull of $\chi_{\mathscr{A}}$. We shall call the sequence $\{\Gamma_n\}$ where $\Gamma_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$, a Morris sequence associated with $(\lambda, \Omega, \Lambda)$. Following Morris' approach, we use Morris sequences to replace convex combinations to define convex subfamily $\mathscr{S} \subset \mathscr{A}$ as in [3].

Definition 2.1. A subfamily $\mathscr{S} \subset \mathscr{A}$ is said to be convex if for every $(\lambda, \Omega, \Lambda) \in I \times \mathscr{S} \times \mathscr{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\lambda, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathscr{S}$.

Morris' definition of a convex set function on $\mathscr{A}$ [2] can now be generalized to a convex subfamily:

Definition 2.2. Let $\mathscr{S}$ be a convex subfamily of $\mathscr{A}$. A set function $F: \mathscr{S} \to \mathbb{R}$ is said to be convex if given $(\lambda, \Omega, \Lambda) \in I \times \mathscr{S} \times \mathscr{S}$ and any Morris sequence $\{\Gamma_n\}$ associated with $(\lambda, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathscr{S}$ such that

$$\limsup_{k \to \infty} F(\Gamma_{n_k}) \leqslant \lambda F(\Omega) + (1-\lambda)\, F(\Lambda).$$

For basic properties of convex subfamilies of $\mathscr{A}$ and convex set functions, the reader is referred to [3]. Note that Definition 2.2 is more general than the definitions given for convex set functions in [2]. However, all the results developed in [2] hold for the redefined convex set functions. Definition 2.2 for convex set functions is further justified by the characterization of their epigraphs in Section 3.

We remark that the results in this paper can be extended to the case of σ-finite measure space and extended real value convex set functions without major difficulties; see [4].

3. Epigraphs of Convex Set Functions

For a convex set function $F: \mathscr{S} \to \mathbb{R}$, $F(\Omega) = F(\Lambda)$ if $\chi_\Omega = \chi_\Lambda$ a.e. in L_∞; therefore, F can be regarded as a functional on L_∞ over the set $\chi_{\mathscr{S}}$.

DEFINITION 3.1. Let $F\colon \mathscr{S} \to \mathbb{R}$ be a set function and $\mathscr{S} \subset \mathscr{A}$. The epigraph of F over $\mathscr{S}$, $[F, \mathscr{S}]$, is defined as

$$[F, \mathscr{S}] = \{(r, \Omega) \in \mathbb{R} \times \mathscr{A} : \Omega \in \mathscr{S}, F(\Omega) \leqslant r\}.$$

DEFINITION 3.2. A subset $A \subset \mathbb{R} \times \mathscr{A}$ is said to be convex if given (r, Ω), $(s, \Lambda) \in A$ and $\lambda \in I$, then for every Morris sequence $\{\Gamma_n\}$ associated with $(\lambda, \Omega, \Lambda)$ there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and a sequence $t_k \to \lambda r + (1-\lambda)\, s$ such that $\{(t_k, \Gamma_{n_k})\} \subset A$.

We now may characterize the convexity of set functions by the convexity of their epigraphs.

THEOREM 3.3. *Let $F\colon \mathscr{S} \to \mathbb{R}$ be a set function and $\mathscr{S}$ a convex subfamily of $\mathscr{A}$. Then F is convex if and only if $[F, \mathscr{S}]$ is a convex subset in $\mathbb{R} \times \mathscr{A}$.*

Proof. Assume that F is convex. Let (r, Ω), $(s, \Lambda) \in [F, \mathscr{S}]$ and $\lambda \in I$. Let $\{\Gamma_n\}$ be a Morris sequence associated with $(\lambda, \Omega, \Lambda)$. By convexity of F, there exists a subsequence $\{\Gamma_{n_k}\}$ in $\mathscr{S}$ such that $\limsup_{k\to\infty} F(\Gamma_{n_k}) \leqslant \lambda F(\Omega) + (1-\lambda)\, F(\Lambda)$, hence $\limsup_{k\to\infty} F(\Gamma_{n_k}) \leqslant \lambda r + (1-\lambda)\, s$. Therefore, a subsequence $\{\Gamma_{n_{k_i}}\}$ of $\{\Gamma_{n_k}\}$ can be found for which $F(\Gamma_{n_{k_i}}) \leqslant \lambda r + (1-\lambda)\, s + 1/i$. This shows that $[F, \mathscr{S}]$ is convex in $\mathbb{R} \times \mathscr{A}$.

Conversely, assume that $[F, \mathscr{S}]$ is a convex in $\mathbb{R} \times \mathscr{A}$. Let $(\lambda, \Omega, \Lambda) \in I \times \mathscr{S} \times \mathscr{S}$ be given. Then since $(F(\Omega), \Omega)$, $(F(\Lambda), \Lambda) \in [F, \mathscr{S}]$, for every Morris sequence $\{\Gamma_n\}$ associated with $(\lambda, \Omega, \Lambda)$ there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and a sequence $t_k \to \lambda F(\Omega) + (1-\lambda)\, F(\Lambda)$ such that $(t_k, \Gamma_{n_k}) \in [F, \mathscr{S}]$ for all k by the convexity of $[F, \mathscr{S}]$. It follows that $\limsup F(\Gamma_{n_k}) \leqslant \lambda F(\Omega) + (1-\lambda)\, F(\Lambda)$. This completes the proof.

When there is no danger of ambiguity, we shall identify $\Omega \in \mathscr{A}$ with $\chi_\Omega \in L_\infty$. Let $A \subseteq \mathbb{R} \times \mathscr{A}$, $\bar{A}$ is defined as the w^*-closure of A in $\mathbb{R} \times L_\infty$. The following proposition shows that the w^*-closure preserves the convex sets.

PROPOSITION 3.4. *If A is a convex subset in $\mathbb{R} \times \mathscr{A}$, then $\bar{A}$ is the w^*-closed convex hull of A in $\mathbb{R} \times L_\infty$.*

Proof. It follows directly from the definition of convexity of A that the convex hull of A, conv A, is contained in $\bar{A}$. Consequently, $\bar{A} \subseteq \overline{\operatorname{conv} A} \subseteq \bar{A}$.

PROPOSITION 3.5. *If $\mathscr{S} \subset \mathscr{A}$ is convex, then $\bar{\mathscr{S}}$ is convex in L_∞, hence the w^*-closed convex hull of $\mathscr{S}$.*

Proof. It is clear that $\{0\} \times \mathscr{S}$ is convex in $\mathbb{R} \times \mathscr{A}$ if $\mathscr{S}$ is convex. By Proposition 3.4, $\overline{\{0\} \times \mathscr{S}} = \{0\} \times \bar{\mathscr{S}}$ is convex in $\mathbb{R} \times L_\infty$, it follows that $\bar{\mathscr{S}}$ is convex in L_∞, hence the w^*-closed convex hull of $\mathscr{S}$.

COROLLARY 3.6. $\bar{\mathscr{A}} = \{f \in L_\infty : 0 \leqslant f \leqslant 1\}$.

Proof. Write $B^+ = \{f \in L_\infty : 0 \leqslant f \leqslant 1\}$. As shown in [5, Theorem 5.5], B^+ is the w^*-compact and $\mathscr{A}$ is the set of all extreme points of B^+, B^+ is the w^*-closed convex hull of $\mathscr{A}$ by the Krein–Milman theorem. Therefore, by Corollary 3.5 $\bar{\mathscr{A}} = B^+$.

Remark 3.7. It follows from Corollary 3.6 that $\bar{\mathscr{A}}$ is nowhere dense in L_∞ with respect to the w^*-topology, since any non-empty w^*-open set in L_∞ is unbounded.

For $f \in B^+$, we denote $\mathscr{N}(f)$ the family of all w^*-neighborhood of f in B^+. Note that since B^+ is w^*-compact and L_1 is separable by assumption, B^+ is metrizable.

DEFINITION 3.8. Let $F: \mathscr{S} \to \mathbb{R}$ be a set function. The w^*-lower semicontinuous hull of F, or the closure of F, is the functional $\bar{F}$ on L_∞ over $\bar{\mathscr{S}}$ defined by

$$\bar{F}(f) = \sup_{V \in \mathscr{N}(f)} \inf_{A \in V \cap \mathscr{S}} F(A) \qquad \text{for } f \in \bar{\mathscr{S}}.$$

F is said to be w^*-lower semicontinuous or simply w^*-l.s.c. if $F(\Omega) = \bar{F}(\Omega)$ for $\Omega \in \mathscr{S}$.

Similarly, we define the w^*-upper semicontinuous or simply w^*-u.s.c. hull $\hat{F}$ of F and the w^*-upper semicontinuity of F. F is said to be w^*-continuous if F is both w^*-l.s.c. and w^*-u.s.c. It is standard to show by definitions that $\bar{F}(\Omega) \leqslant F(\Omega) \leqslant \hat{F}(\Omega)$ for $\Omega \in \mathscr{S}$, and $\bar{F}$ ($\hat{F}$, respectively) is w^*-l.s.c. (w^*-u.s.c., respectively); and if F is w^*-continuous then $\bar{F} \equiv \hat{F}$ on $\bar{\mathscr{S}}$ and $\bar{F}$ is the unique w^*-continuous extension of F.

PROPOSITION 3.9. *Let* $F: \mathscr{S} \to \mathbb{R}$ *be a set function. Then* $\overline{[F, \mathscr{S}]} = [\bar{F}, \bar{\mathscr{S}}]$.

Proof. Since $\bar{F}(\Omega) \leqslant F(\Omega)$ for $\Omega \in \mathscr{S}$, $[F, \mathscr{S}] \subseteq [\bar{F}, \bar{\mathscr{S}}]$. $[\bar{F}, \bar{\mathscr{S}}]$ is closed, since $\bar{F}$ is w^*-l.s.c.; hence $\overline{[F, \mathscr{S}]} \subseteq [\bar{F}, \bar{\mathscr{S}}]$. Conversely, let $(r, f) \in [\bar{F}, \bar{\mathscr{S}}]$. Then $\bar{F}(f) = \sup_{V \in \mathscr{N}(f)} \inf_{A \in V \cap \mathscr{S}} F(A) \leqslant r$. Now since $f \in \bar{\mathscr{S}} \subset B^+$, and B^+ is metrizable, a sequence $\{A_n\}$ can be found in $\mathscr{S}$ such that $\chi_{A_n} \to^{w^*} f$ and $F(A_n) \leqslant r + 1/n$. This shows that $(r, f) \in \overline{[F, \mathscr{S}]}$, and hence $\overline{[F, \mathscr{S}]} = [\bar{F}, \bar{\mathscr{S}}]$.

COROLLARY 3.10. *Let F be a convex set function defined on a convex subfamily* $\mathscr{S} \subset \mathscr{A}$. *Then* $\bar{F}: \bar{\mathscr{S}} \to \mathbb{R}$ *is a convex functional on* L_∞ *over* $\bar{\mathscr{S}}$.

Proof. $[F, \mathscr{S}]$ is convex in $\mathbb{R} \times \mathscr{A}$ by Theorem 3.3. $\overline{[F, \mathscr{S}]}$ is convex in $\mathbb{R} \times L_\infty$ by Proposition 3.4. Now by Proposition 3.9 $[\bar{F}, \bar{\mathscr{S}}] = \overline{[F, \mathscr{S}]}$, hence convex. It follows that $\bar{F}$ is convex on L_∞ over $\bar{\mathscr{S}}$.

COROLLARY 3.11. *Let $\mathscr{S}$ be a convex subfamily of $\mathscr{A}$ and let $F: \mathscr{S} \to \mathbb{R}$ be a w^*-continuous function. Then F is convex if and only if $\bar{F}$ is convex.*

Proof. It follows directly from Proposition 3.10 and the definitions.

By Remark 3.7 for any $\mathscr{S} \subset \mathscr{A}$, $\bar{\mathscr{S}}$ has empty relative w^*-interior, hence for any $F: \mathscr{S} \to \mathbb{R}$, $[\bar{F}, \bar{\mathscr{S}}]$ has empty relative w^*-interior since $v([\bar{F}, \bar{\mathscr{S}}])$, the linear variety generated by $[\bar{F}, \bar{\mathscr{S}}]$, is $\mathbb{R} \times v(\bar{\mathscr{S}})$.

COROLLARY 3.12. *Let $F: \mathscr{S} \to \mathbb{R}$ be a convex set function. If $\bar{\mathscr{S}}$ has non-empty relative interior* (w.r.t. *the L_∞-norm topology*), *and F is w^*-continuous, then $[\bar{F}, \bar{\mathscr{S}}]$ has non-empty relative interior.*

Proof. By Corollary 3.11, $\bar{F}$ is a w^*-continuous convex functional on L_∞ over $\bar{\mathscr{S}}$. Since w^*-continuity implies the L_∞-norm continuity, $[\bar{F}, \bar{\mathscr{S}}]$ has non-empty relative interior by [6, Section 7.9, Proposition 1].

4. FENCHEL DUALITY THEOREM

Since the w^*-closure of the epigraph of a convex set function is equal to the epigraph of a w^*-l.s.c. convex functional on L_∞, duality principles of Fenchel type for set functions can be developed if separating hyperplanes for epigraphs can be found. We shall use the functional notation $\langle f, \chi_\Omega \rangle$ for $\int_\Omega f\, dm$.

THEOREM 4.1. *Let F, G be set functions which are absolutely continuous* w.r.t. m, i.e., *for $\Omega \in \mathscr{A}$, $F(\Omega) = \langle f_0, \chi_\Omega \rangle$ and $G(\Omega) = \langle g_0, \chi_\Omega \rangle$ for some f_0, $g_0 \in L_1(X, \mathscr{A}, m)$. Then* $\min_{\Omega \in \mathscr{A}}[F(\Omega) - G(\Omega)] = \max_{f \in L_1}[G^*(f) - F^*(f)]$, where

$$G^*(f) = \inf_{\Omega \in \mathscr{A}} [G(\Omega) - \langle f, \chi_\Omega \rangle], \qquad F^*(f) = \sup_{\Omega \in \mathscr{A}} [\langle f, \chi_\Omega \rangle - F(\Omega)],$$

and the minimum on the left is achieved by $\Omega_* = (f_0 - g_0)^{-1}((-\infty, 0])$ and the maximum on the right is achieved by any f_* for which $(f_* - g_0)^{-1}((-\infty, 0]) = (f_0 - f_*)^{-1}((-\infty, 0])$, in particular, f_* can be chosen as $(f_0 + g_0)/2$.

Proof. Since $\min_{\Omega \in \mathscr{A}}[F(\Omega) - G(\Omega)] = \min_{\Omega \in \mathscr{A}} \int_\Omega (f_0 - g_0)\, dm$, $\Omega_* = (f_0 - g_0)^{-1}((-\infty, 0])$ is clearly a solution. Let $f_* = (f_0 + g_0)/2$ and $\mu = \min_{\Omega \in \mathscr{A}}[F(\Omega) - G(\Omega)]$. Note that $\bar{F}(f) = \langle f_0, f \rangle$ and $\bar{G}(f) = \langle g_0, f \rangle$ both are w^*-continuous linear functionals on L_∞ and $\inf_{\Omega \in \mathscr{A}}[F(\Omega) - G(\Omega)] = \inf_{f \in D}[\bar{F}(f) - \bar{G}(f)]$, where $D = \{f \in L_\infty : \langle f_0 - g_0, f \rangle \geq \langle f_0 - g_0, \chi_\Omega \rangle\}$ which is a w^*-neighborhood of χ_{Ω_*} and contains B^+. Therefore, a w^*-closed separating hyperplane is guaranteed for

$[\bar{F}, D]$ and $[\bar{G}, D]$. By straightforward calculations, a separating hyperplane is given by

$$\left\{(r,f)\in\mathbb{R}\times L_\infty\colon \langle f,f_*\rangle - r = \int_{\Omega_*}\frac{f_0-g_0}{2}\,dm\right\}.$$

It follows from a standard argument that the equality in the theorem is established.

Remark 4.2. In [7], a definition of epigraphs for convex set functions was given, however, the proof that the epigraph of a convex set function is convex in [7, Proposition 1] is incorrect. Furthermore, the main theorem, the Fenchel duality theorem in [7], is vacuous, since both the epigraphs have empty relative w^*-interiors by Remark 3.7, and all the examples given in [7] are subsumed in Theorem 4.1 which is a duality theorem of Fenchel type only for a very special class of set functions. To derive a general Fenchel duality theorem for set functions we need to define the conjugate functionals for set functions on $(L_\infty)^*$ instead of L_1, the w^*-dual of L_∞. Note that $(L_\infty)^*$ can be characterized as finitely additive set functions [8].

DEFINITION 4.3. Let $\mathscr{T}$ be a convex subfamily of $\mathscr{A}$. A set function $G\colon \mathscr{T}\to\mathbb{R}$ is said to be concave if $-G$ is convex. Define $[G,\mathscr{T}]=\{(r,\Omega)\in\mathbb{R}\times\mathscr{A}\colon r\leqslant G(\Omega)\}$, the epigraph of G over $\mathscr{T}$.

The set $[G,\mathscr{T}]$ is convex in $\mathbb{R}\times\mathscr{A}$ and $\overline{[G,\mathscr{T}]}=[\hat{G},\bar{\mathscr{T}}]$ is convex in $\mathbb{R}\times L_\infty$. All the other results for convex set functions have direct extensions for concave set functions.

DEFINITION 4.4. Let F (G, respectively) be a convex (concave, respectively) function defined on a convex subfamily $\mathscr{S}$ ($\mathscr{T}$, respectively) of $\mathscr{A}$. Define $\mathscr{S}_F^*=\{x^*\in(L_\infty)^*\colon \sup_{\Omega\in\mathscr{S}}[\langle x^*,\chi_\Omega\rangle - F(\Omega)<\infty\}$ and $\mathscr{T}_G^*=\{x^*\in(L_\infty)^*\colon \inf_{\Omega\in\mathscr{T}}[\langle x^*,\chi_\Omega\rangle - G(\Omega)>-\infty\}$. Define F^*, the conjugate functional of F on $\mathscr{S}_F^*$ to be $F^*(x^*)=\sup_{\Omega\in\mathscr{S}}[\langle x^*,\chi_\Omega\rangle - F(\Omega)]$, $x^*\in\mathscr{S}_F^*$; and define G^*, the conjugate functional of G on $\mathscr{T}_G^*$ to be $G^*(x^*)=\inf[\langle x^*,\chi_\Omega\rangle - G(\Omega)]$, $x^*\in\mathscr{T}_G^*$.

THEOREM 4.5 (Fenchel duality theorem). *Assume that F and G are, respectively, convex and concave set functions on a convex subfamily $\mathscr{D}$ of a σ-algebra $\mathscr{A}$ of all measurable subsets in an atomless finite measure space $(X,\mathscr{A},m)$ with $L_1(X,\mathscr{A},m)$ separable. Assume that $\bar{\mathscr{D}}$ contains relative interior points and that either F or G is w^*-continuous. Suppose further that $\mu=\inf_{\Omega\in\mathscr{D}}\{F(\Omega)-G(\Omega)\}$ is finite. Then*

$$\mu=\inf_{\Omega\in\mathscr{D}}\{F(\Omega)-G(\Omega)\}=\max_{x^*\in\mathscr{D}_F^*\cap\mathscr{D}_G^*}\{G^*(x^*)-F^*(x^*)\}$$

where the maximum on the right is achieved by some $x_0^* \in \mathscr{D}_F^* \cap \mathscr{D}_G^*$. If the infimum on the left is achieved by some $\Omega_0 \in \mathscr{D}$, then $\max_{\Omega \in \mathscr{D}}[\langle x_0^*, \chi_\Omega\rangle - F(\Omega)] = \langle x_0^*, \chi_{\Omega_0}\rangle - F(\Omega_0)$ and $\min_{\Omega \in \mathscr{D}}[\langle x_0^*, \chi_\Omega\rangle - G(\Omega)] = \langle x_0^*, \chi_{\Omega_0}\rangle - G(\Omega_0)$.

Proof. By definition, for all $x^* \in \mathscr{D}_F^* \cap \mathscr{D}_G^*$, $\chi_\Omega \in \mathscr{D}$,

$$F^*(x^*) \geqslant \langle x^*, \chi_\Omega\rangle - F(\Omega) \qquad \text{and} \qquad G^*(x^*) \leqslant \langle x^*, \chi_\Omega\rangle - G(\Omega).$$

Thus, $F(\Omega) - G(\Omega) \geqslant G^*(x^*) - F^*(x^*)$ and hence $\inf_{\Omega \in \mathscr{D}}\{F(\Omega) - G(\Omega)\} \geqslant \sup_{x^* \in \mathscr{D}_F^* \cap \mathscr{D}_G^*}\{G^*(x^*) - F^*(x^*)\}$. Therefore, it suffices to show that $\mu = G^*(x_0^*) - F^*(x_0^*)$ for some $x_0^* \in \mathscr{D}_F^* \cap \mathscr{D}_G^*$. By assumption, either F or G, say F is w^*-continuous (the proof proceeds similarly if G is w^*-continuous). We claim that $[\hat{G}, \bar{\mathscr{D}}]$ does not contain any relative interior point of $[\bar{F} - \mu, \bar{\mathscr{D}}]$. Suppose not, and let (r, f) be in the intersection of $[\hat{G}, \bar{\mathscr{D}}]$ and the relative interior of $[\bar{F} - \mu, \bar{\mathscr{D}}]$. Then there exists an $\varepsilon > 0$ such that $\bar{F}(f) - \mu < r - \varepsilon$, and a sequence $\{\Omega_n\}$ in $\mathscr{D}$ can be found for which $\chi_{\Omega_n} \to^{w^*} f$ and $\limsup G(\Omega_n) \geqslant r$ since $(r, f) \in [\hat{G}, \bar{\mathscr{D}}] = \overline{[G, \mathscr{D}]}$. Now since $\bar{F}$ is w^*-continuous, $\lim_{n \to \infty} F(\Omega_n) = \bar{F}(f)$; hence for sufficiently large k we have $G(\Omega_k) \geqslant r - \varepsilon > F(\Omega_k) - \mu$ which contradicts the definition of μ.

By Corollary 3.12, $[\bar{F} - \mu, \bar{\mathscr{D}}]$ has non-empty relative interior; it follows that there is a closed hyperplane in $\mathbb{R} \times L_\infty$ separating $[\bar{F} - \mu, \bar{\mathscr{D}}]$ and $[\hat{G}, \bar{\mathscr{D}}]$, hence $[F - \mu, \mathscr{D}]$ and $[G, \mathscr{D}]$. Since $\bar{\mathscr{D}}$ has relative interior, this hyperplane cannot be vertical, it then can be represented as $\{(r, f) \in \mathbb{R} \times L_\infty \colon \langle f, x_0^*\rangle - r = c\}$ for some $x_0^* \in (L_\infty)^*$ and $c \in \mathbb{R}$. Now since $[G, \mathscr{D}]$ lies below this hyperplane and is arbitrarily close to it, we have $c = \inf_{\Omega \in \mathscr{D}}\{\langle x_0^*, \chi_\Omega\rangle - G(\Omega)\} = G^*(x_0^*)$. Likewise, $c = \sup_{\Omega \in \mathscr{D}}\{\langle x_0^*, \chi_\Omega\rangle - (F(\Omega) - \mu)\} = F^*(x_0^*) + \mu$. Thus $\mu = G^*(x_0^*) - F^*(x_0^*)$. If the infimum μ on the left is attainted by some $\Omega_0 \in \mathscr{D}$, the set $[F - \mu, \mathscr{D}]$ and $[G, \mathscr{D}]$ have the point $(G(\Omega_0), \Omega_0)$ in common and this point lies in the separating hyperplane; thus the last two equalities in the theorem are proved.

Remark 4.6. Let F and G be, respectively, convex and concave set functions on the convex subfamilies $\mathscr{S}$ and $\mathscr{T}$. Write $\mathscr{D} = \mathscr{S} \cap \mathscr{T}$. Assume F (or G, respectively) is w^*-continuous on $\mathscr{S}$ ($\mathscr{T}$, respectively), then Theorem 4.5 holds if $\mathscr{S}_F^*$ ($\mathscr{T}_G^*$, respectively) replaces $\mathscr{D}_F^*$ ($\mathscr{D}_G^*$, respectively).

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